Unit 4: Partial Differential Equations and its applications

A) By elimination of arbitrary constant.

$F(x, y, z, a, b) = 0$

diff. p. w.r.t. x & y

$\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

& it's sol^n $F(x, y, z, p, q) = 0$

B) By eliminating arbitrary constant / Function.

$\phi(u, v) = 0$

$Pp + Qq = R$ Which is the partial diff. eq^n of First order in p, q.

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $r = \frac{\partial^2 z}{\partial x^2}$ $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

$t = \frac{\partial^2 z}{\partial y^2}$

Examples

1. Eliminate the arbitrary Function and form the partial differential equation.

$z =$

1) $z = ax + by + ab$

→ Given: $z = ax + by + ab$

diff. w.r.t. x & y

$$\frac{\partial z}{\partial x} = a + 0 + 0 = p$$

$$\boxed{a = p}$$

$$\frac{\partial z}{\partial y} = 0 + b + 0 = q$$

$$\therefore \boxed{q = b}$$

$$\therefore \boxed{z = px + qy + pq}$$

2. $z = ax + a^2y^2 + b$

diff w.r.t. x & y

$$\frac{\partial z}{\partial x} = a + 0 + 0 = p$$

$$\therefore \boxed{a = p}$$

$$\frac{\partial z}{\partial y} = 0 + 2a^2y + 0 = q$$

$$\therefore q = 2a^2y$$

$$\therefore \boxed{q = 2p^2y}$$

It is a solⁿ

3. $z = (x^2 + a)(y^2 + b)$

→ diff. p. w.r.t. x & y

$$\frac{\partial z}{\partial x} = (y^2+b)(2x+0)$$

$$\therefore \quad p = 2x(y^2+b)$$

$$\frac{\partial z}{\partial y} = (x^2+a)(2y)$$

$$\therefore \quad q = 2y(x^2+a)$$

$$\therefore \quad pq = 2x(y^2+b) \times 2y(x^2+a)$$

$$pq = 4xy(x^2+a)(y^2+b)$$

$$\therefore \quad \boxed{pq = 4xyz}$$ — It is sol^n

4. $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

diff. P. w.r.t. x and y

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} + 0$$

$$\boxed{p = \frac{x}{a^2}} \quad \text{i.e.} \quad \boxed{\frac{p}{x} = \frac{1}{a^2}}$$

$$2\frac{\partial z}{\partial y} = 0 + \frac{2y}{b^2}$$

$$\therefore \quad q = \frac{y}{b^2} \quad \text{i.e.} \quad \boxed{\frac{q}{y} = \frac{1}{b^2}}$$

$$\therefore \quad 2z = \frac{p}{x}x^2 + \frac{q}{y}y^2$$

$$\therefore \quad \boxed{2z = px + qy}$$

5. $az + b = a^2x + y$

diff p. w.r.t. x & y

$$a\frac{\partial z}{\partial x} + 0 = a^2 + 0$$

$$a\frac{\partial z}{\partial x} = a^2$$

$$\frac{\partial z}{\partial x} = a$$

$$\boxed{p = a}$$

$$a\frac{\partial z}{\partial y} + 0 = 0 + 1$$

$$a\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{a}$$

$$\therefore \quad q = \frac{1}{a}$$

It's sol^n

$$\therefore \quad \boxed{q = \frac{1}{p}} \qquad \therefore \quad \boxed{pq = 1}$$

6. $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

diff. P. w.r.t. x & y

$$\frac{2x}{a^2}+0+\frac{1}{c^2}2z\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{--- (A)}$$

$$\frac{x}{a^2}=-\frac{z}{c^2}\frac{\partial z}{\partial x}$$

$$\therefore \boxed{\frac{1}{a^2}=-\frac{z}{c^2x}\frac{\partial z}{\partial x}} \quad \text{--- (1)}$$

eq^n (A) diff. w.r.t. x

$$\frac{1}{a^2}+\frac{z}{c^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{1}{a^2}+\frac{z}{c^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial x}\right)^2=0$$

$$-\frac{z}{c^2x}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial x}\right)^2=0$$

$$\frac{1}{c^2}\left[-\frac{z}{x}\frac{\partial z}{\partial x}+z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right]=0$$

$$-\frac{z}{x}\frac{\partial z}{\partial x} + z\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 = 0$$

$$-\frac{z}{x}p + rz - p^2 = 0$$

$$\boxed{p^2 + rz - \frac{pz}{x} = 0} \quad \text{—— (B)}$$

Now,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

diff. P. w.r.t. y

$$0 + \frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$$

$$\frac{y}{b^2} + \frac{z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{—— (C)}$$

$$\frac{y}{b^2} = \frac{-z}{c^2}\frac{\partial z}{\partial y}$$

$$\therefore \boxed{\frac{1}{b^2} = \frac{-z}{c^2 y}\frac{\partial z}{\partial y}} \quad \text{—— (II)}$$

diff. eqⁿ (C) w.r.t. y

$$\frac{1}{b^2} + \frac{z}{c^2}\frac{\partial^2 z}{\partial y^2} + \frac{1}{c^2}\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial y} = 0$$

$$\frac{1}{b^2} + \frac{z}{c^2}\frac{\partial^2 z}{\partial y^2} + \frac{1}{c^2}\left(\frac{\partial z}{\partial y}\right)^2 = 0$$

$$-\frac{z}{c^2 y}\frac{\partial z}{\partial y} + \frac{z}{c^2}\frac{\partial^2 z}{\partial y^2} + \frac{1}{c^2}\left(\frac{\partial z}{\partial y}\right)^2 = 0$$

$$\frac{1}{c^2}\left[-\frac{z}{y}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial z}{\partial y}\right)^2\right] = 0$$

$$-\frac{z}{y}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial z}{\partial y}\right)^2 = 0$$

$$-\frac{z}{y}q + zt + q^2 = 0$$

$$\therefore \boxed{q^2 + zt - \frac{z}{y}q = 0} \quad \text{—— (D)}$$

$\therefore$ Equation (B) and equation (D) are the complete solution of given equation.

B) By eliminating arbitrary Function

Examples

1. $z = F(x+it) + g(x-it)$

Diff. p. w.r.t. x & y.

$$\frac{\partial z}{\partial x} = F'(x+it)(1+0) + g'(x-it)(1-0)$$

$$= F'(x+it) + g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = F''(x+it)\times 1 + g'(x-it)\times 1$$

$$\frac{\partial^2 z}{\partial x^2} = F''(x+it) + g''(x-it) \quad \text{--- (I)}$$

$$\frac{\partial z}{\partial t} = F'(x+it)\frac{d}{dt}(x+it) + g'(x-it)\frac{d}{dt}(x-it)$$

$$= F'(x+it)(0+i) + g'(x-it)(0-i)$$

$$= iF'(x+it) \cancel{+} - ig'(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = iF''(x+it)\frac{d}{dt}(x+it) - ig''(x-it)\frac{d}{dt}(x-it)$$

$$= iF''(x+it)(0+i) - ig''(x-it)(0-i)$$

$$= i^2F''(x+it) + i^2g''(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = -F''(x+it) - g''(x-it) \quad \text{--- (II)}$$

From (I) & (II)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\therefore \quad \boxed{r + t = 0}$$

2. $z = F(x^2 - y^2)$

Diff. p. w.r.t. x & y

$$\frac{\partial z}{\partial x} = F'(x^2 - y^2) \times 2x$$

$$\boxed{p = 2x F'(x^2 - y^2)}$$

$$\frac{\partial z}{\partial y} = F'(x^2 - y^2) \times 2y$$

$$\boxed{q = -2y \; F'(x^2 - y^2)}$$

$$\therefore \quad \frac{p}{q} = \frac{F'(x^2 - y^2)\, 2x}{-F'(x^2 - y^2)\, 2y}$$

$$\therefore \quad \frac{p}{q} = \frac{x}{-y}$$

$$\therefore \quad -py = qx$$

$$\therefore \quad \boxed{qx + py = 0}$$

3. $z = x + y + F(xy)$

Diff. p. w.r.t. x & y.

$$\frac{\partial z}{\partial x} = 1 + 0 + F'(xy)\,y$$

$$p = 1 + y\,F'(xy) \qquad \therefore\ p - 1 = y\,F'(xy) \quad \text{— (I)}$$

$$\frac{\partial z}{\partial y} = 0 + 1 + F'(xy)\,x$$

$$q = 1 + x\,F'(xy)$$

$$q - 1 = x\,F'(xy) \quad \text{— (II)}$$

$$\frac{p-1}{q-1} = \frac{y\,F'(xy)}{x\,F'(xy)}$$

$$\frac{p-1}{q-1} = \frac{y}{x}$$

$$(p-1)x = y(q-1)$$

$$px - x = qy - y$$

$$\boxed{px - qy - x + y = 0}$$

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Lagrange's Linear Equation

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Its solution $\phi(u,v)=0$

Example-1

1. $\frac{y^2 z}{x} p + xzq = y^2$

→ The given eq^n is in the langrange's form
$Pp + Qq = R$

$P = \frac{y^2 z}{x}$ $\quad Q = xz$ $\quad R = y^2$

1. Auxiliary eq^n.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

taking 1^{st} two ratio.

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2\,dx = y^2\,dy$$

On integration.

$$\int x^2\,dx = \int y^2\,dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3}{3} - \frac{y^3}{3} = c$$

$$x^3 - y^3 = 3c = u \quad \text{— (I)}$$

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$$x\,dx = \cancel{7}\, z\,dz$$

On integration

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$\frac{x^2}{2} - \frac{z^2}{2} = c$$

$$x^2 - z^2 = 2c = v \quad \text{---} (11)$$

From (1) & (11)

Its sol^n

$$\phi(u, v) = 0$$

$$\boxed{\phi(x^3 - y^3, x^2 - z^2) = 0}$$

2. $P\cancel{z} - qz = z^2 + (x+y)^2$

→ The given eq^n is in Lagrange's Form

$$Pp + Qq = R$$

$$P = z, \quad Q = -z, \quad R = z^2 + (x+y)^2$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$dx = -dy$$

On integration...

$$\int 1\, dx = -\int 1\, dy$$

$$x = -y + c$$

$$x + y = c = u = a$$

$$\frac{dx}{z} = \frac{dz}{z^2+a^2}$$

$$dx = \frac{z}{z^2+a^2} dz$$

On integration

$$\int 1\,dx = \int \frac{z}{z^2+a^2} dz$$

$$\int 1\,dx = \frac{1}{2}\int \frac{2z}{z^2+a^2} dz$$

$$x = \frac{1}{2}\log(z^2+a^2) + c$$

$$x - \frac{1}{2}\log(z^2+a^2) = c = v$$

Its solution,

$$\phi(u, v) = 0$$

$$\therefore \boxed{\phi\left(x+y,\ x - \tfrac{1}{2}\log\left(z^2+(x+y)^2\right)\right) = 0}$$

3. $P + 3q = 5z + \tan(y-3x)$

→ The given equation is in Lagrange's Form

$$Pp + Qq = R$$

$P = 1 \quad , \quad Q = 3 \quad , \quad R = 5z + \tan(y-3x)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

$$\frac{dx}{1} = \frac{dy}{3}$$

$$3\,dx = dy$$

on integration

$$\int 3\,dx = \int 1\,dy$$

$$3x = y + c$$

$$y - 3x = -c = u = a$$

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)}$$

$$dx = \frac{1}{5z + \tan a}\,dz$$

on integration

$$\int 1\,dx = \int \frac{1}{5z + \tan a}\,dz$$

$$x = \frac{1}{5}\int \frac{5}{5z + \tan a}\,dz$$

$$x = \frac{1}{5}\log(5z + \tan a) + c$$

$$x - \frac{1}{5}\log(5z + \tan a) = c = v$$

Its solution $\phi(u, v) = 0$

$$\boxed{\phi\left(y - 3x,\ x - \tfrac{1}{5}\log\left(5z + \tan(y - 3x)\right)\right) = 0}$$

4. $(y + zx)p - (x + yz)q = x^2 - y^2$

→ The given eq^n is in lagrange's form

$$Pp + Qq = R$$

$\therefore P = (y + zx), \quad Q = -(x + yz), \quad R = (x^2 - y^2)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(y + zx)} = \frac{dy}{-(x + yz)} = \frac{dz}{(x^2 - y^2)}$$

$$\frac{x\,dx + y\,dy - z\,dz}{x(y + zx) - y(x + yz) - z(x^2 - y^2)}$$

$$\frac{x\,dx + y\,dy - z\,dz}{xy + x^2 z - xy - y^2 z - x^2 z + y^2 z} = 0$$

$$x\,dx + y\,dy - z\,dz = 0$$

on integration

$$\int x\,dx + \int y\,dy - \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = 2c = u$$

$$\frac{y\,dx + x\,dy + dz}{y(y+zx) - x(x+yz) + (x^2 - y^2)}$$

$$\frac{y\,dx + x\,dy + dz}{y^2 + xyz - x^2 - xyz + x^2 - y^2}$$

$$y\,dx + x\,dy + dz = 0$$

$$\int y\,dx + \int x\,dy + \int 1\,dz = 0$$

$$xy + xy + z = c$$

$$2xy + z = c = v$$

Its sol^n $\phi(u, v) = 0$

$$\boxed{\phi(x^2 + y^2 + z^2,\ 2xy + z) = 0}$$

5. $(x^2-yz)p + (y^2-zx)q = z^2-xy$

→ The given equation is in langrange's form

$$Pp + Qq = R$$

$$P = x^2-yz, \quad Q = (y^2-zx), \quad R = (z^2-xy)$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)}$$

$$\frac{dx-dy}{(x-y)(x+y)+z(x-y)} = \frac{dy-dz}{(y-z)(y+z)+x(y-z)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

on integration

$$\int \frac{1}{x-y}(dx-dy) = \int \frac{1}{y-z}(dy-dz)$$

$$\int \frac{1}{x-y}\, d(x-y) = \int \frac{1}{y-z}\, d(y-z)$$

$$\log(x-y) = \log(y-z) + \log c$$
$$\log(x-y) - \log(y-z) = \log c$$

$$\log\left[\frac{x-y}{y-z}\right] = \log c$$

$$\frac{x-y}{y-z} = c = u \quad \text{—} \quad (I)$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz)+y(y^2-zx)+z(z^2-xy)} = \frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy}$$

$$\frac{x\,dx+y\,dy+z\,dz}{x^3-xyz+y^3-xyz+z^3-xyz} = \frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy}$$

$$\frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx+y\,dy+z\,dz}{(x+y+z)} = dx+dy+dz$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)\,d(x+y+z)$$

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$$\int x\,dx + \int y\,dy + \int z\,dz = \int (x+y+z)\,d(x+y+z)$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c$$

$$(x^2+y^2+z^2) - (x+y+z)^2 = 2c$$

$$x^2+y^2+z^2 - (x^2+y^2+z^2+xy+yz+zx) = 2c$$

$$x^2+y^2+z^2 - x^2 - y^2 - z^2 - xy - yz - zx = 2c$$

$$-xy - yz - zx = 2c$$

$$-(xy+yz+zx) = 2c$$

$$xy + yz + zx = -2c = v$$

Its solution $\phi(u, v) = 0$

$$\therefore \boxed{\phi\left(\frac{x-y}{y-z},\ xy+yz+zx\right) = 0}$$

** 6) $(mz-ny)p + (nx-lz)q = ly-mx$

→

The given eqⁿ is in lagranges form

$\therefore P = mz-ny,\ Q = nx-lz,\ R = ly-mx$

Consider A.E

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{mzx - nxy + nxy - lyz + lyz - mzx}$$

$$x\,dx + y\,dy + z\,dz = 0$$

on integration

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = 2c = u \quad \text{— (1)}$$

$$\frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - lmz + lny - mnx} = 0$$

$$l\,dx + m\,dy + n\,dz = 0$$

on integration

$$\int l\,dx + \int m\,dy + \int n\,dz = 0$$

$$lx + my + nz = c = v$$

Its solⁿ

$$\phi(u, v) = 0$$

$$\boxed{\phi(x^2 + y^2 + z^2,\ lx + my + nz) = 0}$$

7. $x(y^2+z)p \cancel{+} - y(x^2+z)q = z(x^2-y^2)$

$\rightarrow$

The given eq^n is in lagrange's linear eq^n form

$$Pp + Qq = R$$

$P = x(y^2+z)$, $Q = -y(x^2+z)$, $R = z(x^2-y^2)$

Its A.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{\frac{dx}{x} + \frac{dy}{y}}{\frac{x(y^2+z)}{x} + \frac{-y(x^2+z)}{y}} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{\frac{dx}{x} + \frac{dy}{y}}{y^2+z-x^2-z} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{\frac{dx}{x} + \frac{dy}{y}}{-(x^2-y^2)} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

on integration

$$\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$$

$$\log x + \log y + \log z = \log c$$

$$\log (xyz) = \log c$$

$$xyz = c = u \longrightarrow (1)$$

$$\frac{x dx + y dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$\frac{x dx + y dy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - zx^2 + y^2z}$$

$$x dx + y dy - dz = 0$$

on integration

$$\int x\, dx + \int y\, dy - \int 1\, dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c$$

$$x^2 + y^2 - 2z = 2c = v$$

Its soln

$$\phi(u, v) = 0$$

$$\boxed{\phi(xyz,\ x^2+y^2-2z) = 0}$$

8. $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$

→

The given eqn is in the lagranges linear eqn form

$$Pp + Qq = R$$

$$P = x^2-y^2-yz, \quad Q = x^2-y^2-zx, \quad R = z(x-y)$$

Auxilary Eqn,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{R}{z(x-y)}$$

$$\frac{dx-dy-dz}{x^2-y^2-yz-x^2+y^2+zx-zx+yz}$$

$$dx-dy-dz = 0$$

on integration

$$\int 1\,dx - \int 1\,dy - \int 1\,dz = 0$$

$$x-y-z = c = u$$

$$\frac{xdx-ydy}{x(x^2-y^2-yz)-y(x^2-y^2-zx)} = \frac{dz}{z(x-y)}$$

$$\frac{xdx-ydy}{x^3-xy^2-\cancel{xyz}-x^2y+y^3+\cancel{xyz}} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx-ydy}{x^2(x-y)-y^2(x-y)} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx - y\,dy}{(x-y)(x^2-y^2)} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx - y\,dy}{x^2-y^2} = \frac{dz}{z}$$

$$\frac{1}{2}\,\frac{2x\,dx - 2y\,dy}{x^2-y^2} = \frac{dz}{z}$$

on integration

$$\frac{1}{2}\log(x^2-y^2) = \log z + \log c$$

$$\log(x^2-y^2)^{1/2} = \log z + \log c$$

$$\log(x^2-y^2)^{1/2} - \log z = \log c$$

$$\log\left(\frac{(x^2-y^2)^{1/2}}{z}\right) = \log c$$

$$\frac{(x^2-y^2)^{1/2}}{z} = c = v$$

Its Solution

$$\phi(u,v) = 0$$

$$\therefore \boxed{\phi\left[x-y-z,\ \frac{(x^2-y^2)^{1/2}}{z}\right] = 0}$$

Method of Seperation of Variables

Let us consider the partial differential equation.

$$u(x, y) = X(x)\, Y(y) \quad \text{——} \ (\text{I})$$

where $X(x)$ is the function of 'x' alone.
and $Y(y)$ is the function of 'y' alone.

- Differentiate given eq^n (1) partially w.r.t. x and y we find the value of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and also higher derivative
- After that taking the ~~common~~ equal to common constant k/m/n/p. ~~of~~
- After that putting this values in given eq^n and then variable separate after the integration we get the solution of the given differential equation.

Example -

1. Use the method of Separation of Variable to Solve the eq^n

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{given that}$$

$$u(x, 0) = 6e^{-3x}.$$

Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ — (A)

$u(x,0) = 6e^{-3x}$ — (B)

Let ~~u(t) = xt~~

$u(x,y) = X(x)\,T(t)$

$= XT$

diff p. w r t x & t

$$\frac{\partial u}{\partial x} = T\frac{\partial X}{\partial x}$$

$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

put this value in eq^n A

$$T\frac{\partial X}{\partial x} = 2X\frac{\partial T}{\partial t} + XT$$

$$T\frac{\partial X}{\partial x} = X\left(2\frac{\partial T}{\partial t} + T\right)$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = \frac{2}{T}\frac{\partial T}{\partial t} + 1 = m^2 \text{ (say-const)}$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = -m^2 \quad \text{— (1)}$$

$$\frac{2}{T}\frac{\partial T}{\partial t} + 1 = -m^2$$

$$\frac{2}{T}\frac{\partial T}{\partial t} = -m^2 - 1 \quad \text{— (11)}$$

Consider eqⁿ (I)

$$\frac{1}{X}\frac{\partial X}{\partial x} = -m^2$$

$$\frac{1}{X}\partial X = -m^2 \partial x$$

$$\int \frac{1}{X}\partial X = -m^2 \int 1 \partial x$$

$$\log X = -m^2 x + \log c_1$$

$$\log X = \log e^{-m^2 x} + \log c_1$$

$$\log X - \log e^{-m^2 x} = \log c_1$$

$$\log \left[\frac{X}{e^{-m^2 x}}\right] = \log c_1$$

$$\frac{X}{e^{-m^2 x}} = c_1$$

$$\therefore \boxed{X = c_1 e^{-m^2 x}} \quad \text{— (C)}$$

Consider eqⁿ (II)

$$\frac{2}{T}\frac{\partial T}{\partial t} = -m^2 - 1$$

$$\frac{1}{T}\partial T = -\frac{(1+m^2)}{2}\partial t$$

on integration

$$\int \frac{1}{T}\,\partial T = -\frac{(1+m^2)}{2}\int 1\,\partial t$$

$$\log T = -\frac{(1+m^2)}{2}t + \log c_2$$

$$\log T = \log e^{-\frac{(1+m^2)}{2}t} + \log c_2$$

$$\log T - \log c_2 = \log e^{-\frac{(1+m^2)}{2}t}$$

$$\log \frac{T}{c_2} = \log e^{-\frac{(1+m^2)}{2}t}$$

$$\frac{T}{c_2} = e^{-\frac{(1+m^2)}{2}t}$$

$$\boxed{T = c_2 e^{-\frac{(1+m^2)}{2}t}} \quad \text{--- (D)}$$

From eq^n (C) and (D)

$$u(x,y) = XT$$

$$u(x,y) = c_1 e^{-m^2x}\, c_2 e^{-\frac{(1+m^2)}{2}t} \quad \text{--- (E)}$$

we have

$$u(x,0) = 6e^{-3x}$$

$$6e^{-3x} = c_1c_2\, e^{-m^2x}\, e^{-\frac{(1+m^2)}{2}\times 0}$$

$$= c_1c_2\, e^{-m^2x} \times 1$$

$$6e^{-3x} = C_1C_2e^{-m^2x}$$

$$\boxed{C_1C_2 = 6} \quad \text{—} \quad (iii)$$

$$-3 = -m^2$$

$$\therefore \boxed{m^2 = 3}$$

eq^n E becomes

$$u(x,y) = 6e^{-3x} e^{-\frac{(1+3)}{2}t}$$

$$= 6e^{-3x} e^{-2t}$$

$$\boxed{u(x,y) = 6e^{-3x-2t}}$$

2. Use the method of Seperation of Variable to solve the eq^n

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

→ Sol^n :

we have

$$\frac{\partial^2 y}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{—} \quad (I)$$

$$u(x,y) = XY \quad \text{—} \quad (II)$$

diff. Partially w.r.t. x & y

$$\frac{\partial u}{\partial x} = Y\frac{\partial X}{\partial x}$$

diff. w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = X \frac{\partial Y}{\partial y}$$

eq^n (I) becomes

$$Y \frac{\partial^2 X}{\partial x^2} - 2Y \frac{\partial X}{\partial x} + X \frac{\partial Y}{\partial y} = 0$$

dividing by XY

$$\frac{Y}{XY} \frac{\partial^2 X}{\partial x^2} - \frac{2Y}{XY} \frac{\partial X}{\partial x} + \frac{X}{XY} \frac{\partial Y}{\partial y} = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} - \frac{2}{X} \frac{\partial X}{\partial x} + \frac{1}{Y} \frac{\partial Y}{\partial y} = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} - \frac{2}{X} \frac{\partial X}{\partial x} = -\frac{1}{Y} \frac{\partial Y}{\partial y} = -p^2 \quad \text{—— (A)}$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} - \frac{2}{X} \frac{\partial X}{\partial x} = -p^2$$

$$\frac{1}{X} \left[\frac{\partial^2 X}{\partial x^2} - 2 \frac{\partial X}{\partial x} \right] = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} - 2 \frac{\partial X}{\partial x} = -p^2 X$$

$$\frac{\partial^2 X}{\partial x^2} - 2 \frac{\partial X}{\partial x} + p^2 X = 0$$

$$ax^2 + bx + c = 0$$

$$a = 1, \quad b = -2, \quad c = p^2$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times p^2}}{2 \times 1}$$

$$m = \frac{2 \pm \sqrt{4 - 4p^2}}{2}$$

$$m = \frac{2 \pm 2\sqrt{1 - p^2}}{2}$$

$$m = \frac{2(1 \pm \sqrt{1-p^2})}{2}$$

$$m_1 = 1 + \sqrt{1-p^2}$$
$$m_2 = 1 - \sqrt{1-p^2}$$

$$x = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$x = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

From A

$$-\frac{1}{Y}\frac{\partial Y}{\partial y} = -p^2$$

$$\frac{1}{Y}\,\partial Y = p^2\,\partial y$$

on integration

$$\int \frac{1}{Y}\,\partial Y = p^2 \int 1\,\partial y$$

$$\log Y = p^2 y + \log c_3$$

$$\log Y - \log c_3 = \log e^{p^2 y}$$

$$\log \frac{Y}{c_3} = \log e^{p^2 y}$$

$$\frac{Y}{c_3} = e^{p^2 y}$$

$$Y = c_3 e^{p^2 y}$$

It's sol^n is

$$u(x,y) = XY$$

$$= \left(c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}\right) \times c_3 e^{p^2 y}$$

$$\therefore \boxed{u(x,y) = \left(c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{p-p^2})x}\right) \times c_3 e^{p^2 y}}$$

3) $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$ given that $u=0$ at $t=0$

and $\frac{\partial u}{\partial t} = 0$ when $x = 0$

→ we have given

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \quad \text{--- (I)}$$

$$u(x,t) = XT \longrightarrow \text{(II)}$$

diff. w.r.t. t

$$\frac{\partial u}{\partial t} = X \frac{\partial T}{\partial t}$$

diff. w.r.t. x

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial T}{\partial t} \frac{\partial X}{\partial x}$$

eq^n (I) becomes

$$\frac{\partial T}{\partial t} \frac{\partial X}{\partial x} = e^{-t} \cos x$$

$$\frac{1}{e^{-t}} \frac{\partial T}{\partial t} = \frac{\cos x}{\frac{\partial X}{\partial x}} = -p^2 \quad \text{--- (A)}$$

$$\frac{\cos x}{\frac{\partial X}{\partial x}} = -p^2$$

$$\cos x = -p^2 \frac{\partial X}{\partial x}$$

$$\cos x \, \partial x = -p^2 \partial x$$

$$-\frac{\cos x}{p^2} \partial x = \partial x$$

on integration

$$\cancel{-\frac{\cos x}{p^2} \partial x = \partial x}$$

$$-\frac{1}{p^2} \int \cos x \, \partial x = \int 1 \, \partial x$$

$$-\frac{1}{p^2} \sin x = x + C_1$$

$$\boxed{x = -\frac{\sin x}{p^2} - C_1} \text{ — (B)}$$

From eq^n A

$$\frac{1}{e^t} \cdot \frac{\partial T}{\partial t} = -p^2$$

$$1 \, \partial T = -p^2 e^{-t} \partial t$$

on integration

$$\int 1 \, \partial T = -p^2 \int e^{-t} \partial t$$

$$\boxed{T = -p^2 \frac{e^{-t}}{(-1)} + C_2}$$

$\therefore$

$$\boxed{T = p^2 e^{-t} + c_2}$$

$$u(x,t) = XT$$

$$u(x,t) = \left(\frac{-\sin x}{p^2} - c_1\right)\left(p^2 e^{-t} + c_2\right)$$

—— (C)

We have given

$u = 0$ when $t = 0$

$$0 = \left(\frac{-\sin x}{p^2} - c_1\right)\left(p^2 e^0 + c_2\right)$$

$$0 = p^2 + c_2$$

$$\therefore \boxed{c_2 = -p^2}$$

$$u(x,t) = \left(\frac{-\sin x}{p^2} - c_1\right)\left(p^2 e^{-t} - p^2\right)$$

$$= \left(\frac{-\sin x}{p^2} - c_1\right) p^2 \left(e^{-t} - 1\right)$$

$$u(x,t) = \left(\frac{\sin x}{p^2} + c_1\right) p^2 \left(1 - e^{-t}\right)$$

—— (D)

$$\frac{\partial u}{\partial t} = p^2\left(\frac{\sin x}{p^2} + c_1\right)\left(0 - e^{-t}(-1)\right)$$

$$\frac{\partial u}{\partial t} = p^2\left(\frac{\sin x}{p^2} + c_1\right) e^{-t}$$

We have

$$\frac{\partial u}{\partial t} = 0 \quad \text{when} \quad x = 0$$

$$0 = p^2 (0 + c_1) e^{-t}$$

$$0 = p^2 (0 + c_1)$$

$$\boxed{c_1 = 0}$$

$$u(x,t) = \left(\frac{\sin x}{p^2} + 0 \right) p^2 (1 - e^{-t})$$

$$\boxed{u(x,t) = \sin x \, (1 - e^{-t})}$$

4. Solve by method of separation of variable

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \quad \text{given} \quad u(x,0) = 4e^{-x}$$

→ We have

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \quad \text{--- (I)}$$

$$u(x,y) = XY$$

Diff. w.r.t. x & y

$$\frac{\partial u}{\partial x} = Y \frac{\partial X}{\partial x}$$

$$\frac{\partial u}{\partial y} = X \cdot \frac{\partial Y}{\partial y}$$

eq^n I becomes

$$3Y \frac{\partial X}{\partial z} + 2X \frac{\partial Y}{\partial y} = 0$$

Dividing XY

$$\frac{3Y}{XY} \frac{\partial X}{\partial z} + \frac{2X}{XY} \frac{\partial Y}{\partial y} = 0$$

$$\frac{3}{X} \frac{\partial X}{\partial z} + \frac{2}{Y} \frac{\partial Y}{\partial y} = 0$$

$$\frac{3}{X} \frac{\partial X}{\partial z} = -\frac{2}{Y} \frac{\partial Y}{\partial y} = -p^2$$

$$\frac{3}{X} \frac{\partial X}{\partial z} = -p^2, \quad -\frac{2}{Y} \frac{\partial Y}{\partial y} = -p^2$$

$$\frac{1}{X} \partial X = \frac{-p^2}{3} \partial z \quad \text{—(II)}, \quad \frac{1}{Y} \partial Y = \frac{p^2}{2} \partial y \quad \text{—(III)}$$

Consider eq^n (II)

$$\frac{1}{X} \partial X = -\frac{p^2}{3} \partial z$$

On integration

$$\int \frac{1}{X} \partial X = -\frac{p^2}{3} \int 1 \, \partial z$$

$$\log x = -\frac{p^2}{3}x + \log c_1$$

$$\log x - \log c_1 = \log e^{-\frac{p^2 x}{3}}$$

$$\log \frac{x}{c_1} = \log e^{-\frac{p^2 x}{3}}$$

$$\frac{x}{c_1} = e^{-\frac{p^2 x}{3}}$$

$$\therefore \boxed{x = c_1 e^{-\frac{p^2 x}{3}}} \quad \text{— } (A)$$

Consider eq^n (III)

$$\frac{1}{Y}\partial Y = \frac{p^2}{2}\partial y$$

On integration

$$\int \frac{1}{Y}\partial Y = \frac{p^2}{2}\int 1\,\partial y$$

$$\log Y = \frac{p^2}{2}y + \log c_2$$

$$\log Y - \log c_2 = \frac{p^2}{2}y$$

$$\log Y - \log c_2 = \log e^{\frac{p^2}{2}y}$$

$$\log \frac{Y}{C_2} = \log e^{\frac{p^2 y}{2}}$$

$$\frac{Y}{C_2} = e^{\frac{p^2 y}{2}}$$

$$\therefore \boxed{Y = C_2 e^{\frac{p^2 y}{2}}} \quad \text{—— (B)}$$

$$u(x,y) = XY$$

$$= C_1 e^{-\frac{p^2 x}{3}} \, C_2 e^{\frac{p^2 y}{2}} \quad \text{—— (C)}$$

We have

$$u(x,0) = 4e^{-x}.$$

$$4e^{-x} = C_1 C_2 e^{-\frac{p^2 x}{3}} e^{\frac{p^2(0)}{2}}$$

$$= C_1 C_2 e^{-\frac{p^2 x}{3}} e^{0}$$

$$4e^{-x} = C_1 C_2 e^{-\frac{p^2 x}{3}} \times 1$$

$$\boxed{C_1 C_2 = 4} \qquad -\frac{p^2}{3} = -1$$

$$\boxed{p^2 = 3}$$

eqⁿ (C) becomes

$$u(x,y) = 4 e^{-\frac{3}{3}x} e^{\frac{3}{2}y}$$

$$= 4\, e^{-x}\, e^{\frac{3}{2}y}$$

$$\boxed{u(x,y) = 4e^{-x+\frac{3}{2}y}}$$